Matrix difference equations for the supersymmetric Lie algebra *sl*(2,1) and the `off-shell' Bethe ansatz

2000 J. Phys. A: Math. Gen. 33 2579

(http://iopscience.iop.org/0305-4470/33/13/310)

J. Phys. A: Math. Gen. **33** (2000) 2579–2586. Printed in the UK PII: S0305-4470(00)09940-6

Matrix difference equations for the supersymmetric Lie algebra $sl(2,1)$ and the 'off-shell' Bethe ansatz

T Quella†

Institut für theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany

E-mail: quella@physik.fu-berlin.de

Received 4 January 2000

Abstract. Based on the rational R-matrix of the supersymmetric $sl(2,1)$ matrix difference equations are solved by means of a generalization of the nested algebraic Bethe ansatz. These solutions are shown to be of highest weight with respect to the underlying graded Lie algebra structure.

1. Introduction

The supersymmetric t-J model is often considered a candidate for describing high-T_c superconductivity [1–3]. The underlying symmetry is described by the supersymmetric (graded) Lie algebra $sl(2,1)$. Integrable models with supersymmetry have been discussed in [4–9]. This paper extends the results in [10, 11] on matrix difference equations and a generalized version of the algebraic Bethe ansatz for ordinary or quantum groups to this supersymmetric Lie algebra. We recall that matrix difference equations play an important role in mathematical physics (see, e.g., [12–15]). In particular, in the context of quantum integrable field theories they provide solutions of the formfactor equations, which can be used to calculate correlation functions [16]. This type of matrix difference equations can also be considered as a discrete version [17] of a Knizhnik–Zamolodchikov system [18].

The conventional algebraic Bethe ansatz is used to solve the eigenvalue problem of a Hamiltonian in a way closely related to the underlying symmetry of the considered model (see, e.g., [19]). One constructs the eigenvectors as highest-weight vectors of the corresponding irreducible representations either of the ordinary Lie algebra or the q-deformed analogue, the quantum group. By this construction one encounters 'unwanted' terms. The eigenvalue equation is fulfilled if all of these 'unwanted' terms vanish, which leads to the so-called Bethe ansatz equations.

The 'off-shell' Bethe ansatz [10, 11, 17, 20, 21] is used to solve matrix differential or difference equations. The solution is represented as an integral or a sum over some lattice (an integral of Jackson type). The 'unwanted' terms arising in this case do not vanish due to the Bethe ansatz equations but they sum up to zero under the integral or sum. This modification of the Bethe ansatz was originally introduced to solve Knizhnik–Zamolodchikov equations

† Present address: Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm, Germany. E-mail: quella@aei-potsdam.mpg.de

0305-4470/00/132579+08$30.00

[20]. It has also been applied to the quantization of dimensionally reduced gravity [22] in this connection.

Let $f^{1\cdots n}(\underline{x}) : \mathbb{C}^n \to V^{1\cdots n} = \bigotimes_{j=1}^n \mathbb{C}^3$ be a vector-valued function with the following symmetry property:

$$f^{\cdots ij\cdots}(\ldots, x_i, x_j, \ldots) = R_{ji}(x_j - x_i) f^{\cdots ji\cdots}(\ldots, x_j, x_i, \ldots) \tag{1}$$

where R is the $sl(2,1)$ R-matrix (see below). We consider the set of matrix difference equations

$$f^{1\cdots n}(x_1, \ldots, x_i + \xi, \ldots, x_n) = Q_{1\cdots n}(\underline{x}|i) f^{1\cdots n}(\underline{x}) \qquad (i = 1, \ldots, n) \tag{2}$$

with an arbitrary shift parameter ξ and some sort of generalized transfer matrix $Q_{1\cdots n}(\underline{x}|i)$ which is invariant under $sl(2,1)$. Functions satisfying (1) and (2) will be called R-symmetric and Q-periodic, respectively.

2. Matrix difference equation and generalized nested Bethe ansatz

Let $V^{1\cdots n} = V_1 \otimes \cdots \otimes V_n$ be the tensor product of n isomorphic vector spaces $V_i = \mathrm{Span}\{|1\rangle, |2\rangle, |3\rangle\} \cong \mathbb{C}^3$. The states $|1\rangle$ and $|2\rangle$ are supposed to be bosonic, while $|3\rangle$ is fermionic [3]. For later convenience we also define the reduced vector spaces $\tilde{V}_i = \mathrm{Span}\{|2\rangle, |3\rangle\} \cong \mathbb{C}^2$ and $\tilde{V}^{1\cdots m} = \tilde{V}_1 \otimes \cdots \otimes \tilde{V}_m$. Vectors in $V^{1\cdots n}$ will be denoted by $f^{1\cdots n} \in V^{1\cdots n}$. Analogously, vectors in the reduced spaces are, in addition, marked with a tilde: $\tilde{f}^{1\cdots m} \in \tilde{V}^{1\cdots m}$. Matrices acting in $V^{1\cdots n}$ are denoted by index subscripts $Q_{1\cdots n} : V^{1\cdots n} \to V^{1\cdots n}$.

As usual, the R-matrix will depend on a spectral parameter θ. This matrix $R_{ij}(\theta) : V_i \otimes V_j \to V_j \otimes V_i$ is of the form [23]

$$R_{ij}(\theta) = b(\theta)\Sigma_{ij} + c(\theta)P_{ij} \tag{3}$$

where $P_{ij} : |\alpha\rangle \otimes |\beta\rangle \mapsto |\alpha\rangle \otimes |\beta\rangle$ is the permutation operator and

$$\Sigma_{ij} : |\alpha\rangle \otimes |\beta\rangle \mapsto \sigma_{\alpha\beta}|\beta\rangle \otimes |\alpha\rangle = \begin{cases} -|\beta\rangle \otimes |\alpha\rangle & |\alpha\rangle = |\beta\rangle = |3\rangle \\ |\beta\rangle \otimes |\alpha\rangle & \text{otherwise.} \end{cases}$$

The statistics factor $\sigma_{\alpha\beta} = \pm 1$ takes the fermionic character of the state $|3\rangle$ into account. It has the value -1 if and only if both states are fermionic, i.e. $\alpha = \beta = 3$. The functions $b(\theta)$ and $c(\theta)$ have the form

$$b(\theta) = \frac{\theta}{\theta + K} \qquad c(\theta) = \frac{K}{\theta + K}$$

with an arbitrary constant K. For later use we define the function $w(\theta) = -b(\theta) + c(\theta)$. It is easy to check that $R(\theta)$ is unitary and satisfies the Yang–Baxter equation:

$$R_{ab}(\theta)R_{ba}(-\theta) = \mathbf{1} \qquad \text{and} \qquad R_{12}(\theta_{12})R_{13}(\theta_{13})R_{23}(\theta_{23}) = R_{23}(\theta_{23})R_{13}(\theta_{13})R_{12}(\theta_{12}) \tag{4}$$

where $\theta_{ij} = \theta_i - \theta_j$.

Next we introduce different kinds of monodromy matrices which prove to be useful in the following. The monodromy matrix

$$T_{1\cdots n,a}(\underline{x}|u) = R_{1a}(x_1 - u) \cdots R_{na}(x_n - u)$$

is an operator $V^{1\cdots n}\otimes V_a \to V_a\otimes V^{1\cdots n}$. The vector spaces $V^{1\cdots n}$ and V_a are called the quantum and auxiliary space, respectively. As usual we will consider this operator as a matrix

$$T_{1\cdots n,a} = \begin{pmatrix} A & B_2 & B_3 \\ C^2 & D_2^2 & D_3^2 \\ C^3 & D_2^3 & D_3^3 \end{pmatrix}$$

over the auxiliary space with operators in the quantum space as entries. As a consequence of (4) the monodromy matrix fulfils the Yang–Baxter algebra relation

$$R_{ab}(u-v)T_{1\cdots n,b}(\underline{x}|v)T_{1\cdots n,a}(\underline{x}|u) = T_{1\cdots n,a}(\underline{x}|u)T_{1\cdots n,b}(\underline{x}|v)R_{ab}(u-v). \quad (5)$$

Following [10] we also introduce another set of modified monodromy matrices for $i = 1, \ldots, n$ given as

$$T^{Q}_{1\cdots n,a}(\underline{x}|i) = R_{1a}(x_1 - x_i)\cdots R_{i-1a}(x_{i-1}-x_i)P_{ia}R_{i+1a}(x_{i+1}-x_i')\cdots R_{na}(x_n - x_i') \quad (6)$$

where $\underline{x}\,' = \underline{x} + \xi\underline{e}_i$. In the same way as above they should be considered as matrices in the auxiliary space. This new type of monodromy matrix satisfies the two mixed Yang–Baxter relations

$$T^{Q}_{1\cdots n,a}(\underline{x}|i)T_{1\cdots n,b}(\underline{x}|u)R_{ab}(x_i'-u) = R_{ab}(x_i-u)T_{1\cdots n,b}(\underline{x}\,'|u)T^{Q}_{1\cdots n,a}(\underline{x}|i) \quad (7)$$

$$T_{1\cdots n,a}(\underline{x}\,'|u)T^{Q}_{1\cdots n,a}(\underline{x}|i)R_{ab}(u-x_i') = R_{ab}(u-x_i)T^{Q}_{1\cdots n,b}(\underline{x}|i)T_{1\cdots n,a}(\underline{x}|u). \quad (8)$$

For $i = n$ the modified monodromy matrix is the same as the ordinary one.

We want to encode the fermionic nature of the state $|3\rangle$ in such a way that $sl(2,1)$ appears naturally. To do so we define an additional monodromy matrix

$$[T^{\star}{}_{1\cdots n,a}(\underline{x}|u)]^{\beta,\{\nu\}}_{\alpha,\{\mu\}} = \sigma_{\alpha\beta}\sigma_{\beta\nu_1}\cdots\sigma_{\beta\nu_n}[T_{1\cdots n,a}(\underline{x}|u)]^{\beta,\{\nu\}}_{\alpha,\{\mu\}} \quad (9)$$

where the quantum space indices are collected in the notation $\{\nu\} = \nu_1, \ldots, \nu_n$. This definition is easily extended to a modified version as before. The shift operator is defined by

$$Q_{1\cdots n}(\underline{x}|i) = \mathrm{tr}_a\, T^{\star Q}_{1\cdots n,a}(\underline{x}|i) = A^{Q}_{1\cdots n,a}(\underline{x}|i) + \sum_{\alpha=2,3}\left[D^{\star Q}_{1\cdots n,a}(\underline{x}|i)\right]^{\alpha}_{\alpha} \quad (10)$$

which is obviously closely related to usual transfer matrices. For all operators just defined there also exists a counterpart in the reduced spaces denoted by a tilde.

Using the Yang–Baxter relations given above we derive in a straightforward way the commutation relations

$$B_i(\underline{x}|u_2)B_j(\underline{x}|u_1) = B_{j'}(\underline{x}|u_1)B_{i'}(\underline{x}|u_2)R^{i'j'}_{ji}(u_1-u_2) \quad (11)$$

$$A(\underline{x}|u_2)B_i(\underline{x}|u_1) = \frac{1}{b(u_2-u_1)}B_i(\underline{x}|u_1)A(\underline{x}|u_2) - \frac{c(u_2-u_1)}{b(u_2-u_1)}B_i(\underline{x}|u_2)A(\underline{x}|u_1) \quad (12)$$

$$A^{Q}(\underline{x}|i)B_j(\underline{x}|u) = \frac{1}{b(x_i'-u)}B_j(\underline{x}\,'|u)A^{Q}(\underline{x}|i) - \frac{c(x_i'-u)}{b(x_i'-u)}B^{Q}_j(\underline{x}|i)A(\underline{x}|u) \quad (13)$$

$$\begin{aligned} D^{\star i}{}_j(\underline{x}|u_2)B_k(\underline{x}|u_1) &= \frac{\sigma_{ik}}{b(u_1-u_2)}B_{k'}(\underline{x}|u_1)D^{\star i}{}_{j'}(\underline{x}|u_2)R^{j'k'}_{kj}(u_1-u_2) \\ &\quad -\sigma_{ik}\frac{c(u_1-u_2)}{b(u_1-u_2)}B_j(\underline{x}|u_2)D^{\star i}{}_k(\underline{x}|u_1) \end{aligned} \quad (14)$$

$$\begin{aligned} D^{\star Qj}_{k}(\underline{x}|i)B_l(\underline{x}|u) &= \sigma_{jl}\frac{1}{b(u-x_i)}B_{l'}(\underline{x}\,'|u)D^{\star Qj}_{k'}(\underline{x}|i)R^{k'l'}_{lk}(u-x_i') \\ &\quad -\sigma_{jl}\frac{c(u-x_i)}{b(u-x_i)}B^{Q}_k(\underline{x}|i)D^{\star j}{}_l(\underline{x}|u). \end{aligned} \quad (15)$$

The first terms on the right-hand side of each of these equations are called 'wanted' and the others 'unwanted'. These relations are slightly different from those appearing in the $SU(N)$ case [10] due to the statistics factors σ in the last two equations.

To solve the system of (1) and the matrix difference equations (2) we use the nested so-called 'off-shell' Bethe ansatz [20, 21] with two levels. The first level is quite analogous to the constructions in [10, 11]. Due to the fermionic statistics of state $|3\rangle$ which ensures supersymmetry the second level is different. This problem is solved in the present paper. We write the vector-valued function $f^{1\cdots n}:\mathbb{C}^n\to V^{1\cdots n}$ as a sum of first level Bethe ansatz vectors

$$f^{1\cdots n}(\underline{x})=\sum_{\underline{u}}B_{\beta_m}(\underline{x}|u_m)\cdots B_{\beta_1}(\underline{x}|u_1)\Omega^{1\cdots n}[g^{1\cdots m}(\underline{x}|\underline{u})]^{\beta_1\cdots\beta_m}. \tag{16}$$

The sum is extended over $\underline{u}\in\underline{u}_0-\xi\mathbb{Z}^m\subset\mathbb{C}^m$ (an 'integral of Jackson type', $\underline{u}_0\in\mathbb{C}^m$ arbitrary). The reference state $\Omega^{1\cdots n}$ is given by $\Omega^{1\cdots n}=|1\rangle^{\otimes n}$ and the auxiliary function $g^{1\cdots m}:\mathbb{C}^n\times\mathbb{C}^m\to\tilde{V}^{1\cdots m}$ is defined by $g^{1\cdots m}(\underline{x}|\underline{u})=\eta(\underline{x}|\underline{u})\tilde{f}^{1\cdots m}(\underline{u})$ with $\eta:\mathbb{C}^n\times\mathbb{C}^m\to\mathbb{C}$,

$$\eta(\underline{x}|\underline{u})=\prod_{i=1}^{n}\prod_{j=1}^{m}\psi(x_i-u_j)\prod_{1\leqslant i<j\leqslant m}\tau(u_i-u_j)$$

where the scalar functions $\psi:\mathbb{C}\to\mathbb{C}$ and $\tau:\mathbb{C}\to\mathbb{C}$ satisfy

$$b(x)\psi(x)=\psi(x-\xi)\qquad\frac{\tau(x)}{b(x)}=\frac{\tau(x-\xi)}{b(\xi-x)}. \tag{17}$$

Possible solutions are

$$\psi(x)=\frac{\Gamma(1+K/\xi+x/\xi)}{\Gamma(1+x/\xi)}\qquad\text{and}\qquad\tau(x)=x\frac{\Gamma(x/\xi-K/\xi)}{\Gamma(1+x/\xi+K/\xi)}. \tag{18}$$

They may be multiplied by an arbitrary function which is periodic in ξ.

We prove that $f^{1\cdots n}(\underline{x})$ is R-symmetric and Q-periodic if $\tilde{f}^{1\cdots m}(\underline{u})$ is $\tilde{R}$-symmetric and $\tilde{Q}$-periodic. To compute the action of the shift operator Q on our Bethe ansatz function $f^{1\cdots n}(\underline{x})$ we start from (10) and commute the operators A^Q and $D^{\star Q}$ through all the B-operators to the right where they act on the reference states according to $A^Q(\underline{x}|m)\Omega^{1\cdots n}=\Omega^{1\cdots n}$ and $\left[D^{\star Q}(\underline{x}|m)\right]_{\alpha}^{\alpha'}\Omega^{1\cdots n}=0$. If $\tilde{f}^{1\cdots m}(\underline{u})$ is $\tilde{R}$-symmetric one obtains the representations ($\underline{x}'=\underline{x}+\xi\underline{e}_n$)

$$\begin{aligned}A^{\star Q}(\underline{x}|n)f^{1\cdots n}(\underline{x})&=f^{1\cdots n}(\underline{x}')+\sum_{\underline{u}}\sum_{i=1}^{m}\Lambda_A^{(i)}(\underline{x}|\underline{u})B_{\beta_i}^{Q}(\underline{x}|n)\\&\quad\times B_{\beta_m}(\underline{x}|u_m)\cdots\widehat{B_{\beta_i}(\underline{x}|u_i)}\cdots B_{\beta_1}(\underline{x}|u_1)\Omega^{1\cdots n}\\&\quad\times\eta(\underline{x}|\underline{u})\big[\tilde{f}^{1\cdots mi}(u_1,\ldots,u_m,u_i)\big]^{\beta_1\cdots\beta_m\beta_i}\\ [D^{\star Q}(\underline{x}|n)]_{\alpha}^{\alpha}f^{1\cdots n}(\underline{x})&=\sum_{\underline{u}}\sum_{i=1}^{m}\Lambda_D^{(i)}(\underline{x}|\underline{u})B_{\beta_i}^{Q}(\underline{x}|n)B_{\beta_m}(\underline{x}|u_m)\cdots\widehat{B_{\beta_i}(\underline{x}|u_i)}\cdots B_{\beta_1}(\underline{x}|u_1)\\&\quad\times\Omega^{1\cdots n}\eta(\underline{x}|\underline{u})\big[\tilde{Q}(u_1,\ldots,u_m,u_i|i)\tilde{f}^{1\cdots mi}(u_1,\ldots,u_m,u_i)\big]^{\beta_1\cdots\beta_m\beta_i}.\end{aligned}$$

The hat denotes a factor which is omitted and $\tilde{Q}(u_1,\ldots,u_m,u_i|i)$ is an analogue to the shift operator (10) in the dimensionally reduced space $\tilde{V}^{1\cdots mi}$. The 'wanted' contributions already ensure the validity of (2), so 'unwanted' ones have to sum up to zero. The representation can be obtained as follows: the expression in front of the sum is a consequence of the 'wanted' parts of the commutation relations (11)–(15). To determine the functions $\Lambda_A^{(i)}(\underline{x}|\underline{u})$

and $\Lambda_D^{(i)}(\underline{x}|\underline{u})$ one has to perform the following steps: first move $B_{\beta_i}(\underline{x}|u_i)$ to the front of the B-operators according to (11) and use the $\tilde{R}$-symmetry of $\tilde{f}(\underline{u})$ to absorb them. Then consider the 'unwanted' contributions of (13) and (15), respectively. Now commute the resulting operators $A(\underline{x}|u_i)$ and $D^\star(\underline{x}|u_i)$ to the right and only take the 'wanted' contributions into account. This gives a product of R-matrices and statistics factors in the case of $D^\star$. The action on the reference state is given by $A(\underline{x}|u_i)\Omega^{1\cdots n} = \Omega^{1\cdots n}$ and $[D^\star(\underline{x}|u_i)]_\alpha^{\alpha'}\Omega^{1\cdots n} = \delta_\alpha^{\alpha'}\sigma_{\alpha\alpha'}\prod_{j=1}^n b(x_j - u_i)\Omega^{1\cdots n}$, respectively. By an 'ice rule' for fermions one can regroup the statistics factors. Together with the R-matrices this gives the reduced shift operator $\tilde{Q}(u_1, \ldots, u_m, u_i|i)$. Finally, one obtains

$$\Lambda_A^{(i)}(\underline{x}|\underline{u}) = -\frac{c(x_n' - u_i)}{b(x_n' - u_i)}\prod_{l\neq i}\frac{1}{b(u_i - u_l)}$$

$$\Lambda_D^{(i)}(\underline{x}|\underline{u}) = -\frac{c(u_i - x_n)}{b(u_i - x_n)}\prod_{l\neq i}\frac{1}{b(u_l - u_i)}\prod_{l=1}^n b(x_l - u_i).$$

As already mentioned we have to show that the contributions of the sums cancel. Following the arguments of [10], i.e. using (17) and the relation $c(x)/b(x) = -c(-x)/b(-x)$, one can indeed show that these 'unwanted' contributions vanish after the summation, if $\tilde{f}^{1\cdots m}(\underline{u})$ is $\tilde{Q}$-periodic. The symmetry of $\eta(\underline{x}|\underline{u})$ in the arguments $x_1, \ldots, x_n$ combined with $R_{ij}(\theta)\Omega^{1\cdots n} = \Omega^{1\cdots n}$ implies the R-symmetry of $f^{1\cdots n}(\underline{x})$.

The next step consists of the construction of a function $\tilde{f}^{1\cdots m}(\underline{u})$ which is $\tilde{R}$-symmetric and $\tilde{Q}$-periodic. As above we write

$$\tilde{f}^{1\cdots m}(\underline{u}) = \sum_{\underline{v}}\tilde{B}(\underline{u}|v_k)\cdots\tilde{B}(\underline{u}|v_1)\tilde{\Omega}^{1\cdots m}\tilde{g}(\underline{u}|\underline{v}). \tag{19}$$

The sum is extended over $\underline{v} \in \underline{v}_0 - \xi\mathbb{Z}^k \subset \mathbb{C}^k$ ($\underline{v}_0 \in \mathbb{C}^k$ arbitrary). Here the reference state is given by $\tilde{\Omega}^{1\cdots m} = |2\rangle^{\otimes m}$ and the auxiliary function $\tilde{g} : \mathbb{C}^m \times \mathbb{C}^k \to \mathbb{C}$ reads

$$\tilde{g}(\underline{u}|\underline{v}) = \prod_{i=1}^m\prod_{j=1}^k\psi(u_i - v_j)\prod_{1\leqslant i<j\leqslant k}\tilde{\tau}(v_i - v_j)$$

where $\psi : \mathbb{C} \to \mathbb{C}$ and $\tilde{\tau} : \mathbb{C} \to \mathbb{C}$ satisfy

$$b(x)\psi(x) = \psi(x - \xi) \qquad \frac{\tilde{\tau}(x)}{b(-x)} = \frac{\tilde{\tau}(x - \xi)}{b(\xi - x)}. \tag{20}$$

Possible solutions of (20) are given by (18) and $\tilde{\tau}(x) = x/(x - K)$. Again both functions may be multiplied by an arbitrary function which is periodic in ξ. Note that the supersymmetry has modified the last equation compared to (17).

The Yang–Baxter relations imply the commutation relations

$$\tilde{B}(\underline{u}|v_2)\tilde{B}(\underline{u}|v_1) = w(v_1 - v_2)\tilde{B}(\underline{u}|v_1)\tilde{B}(\underline{u}|v_2)$$

$$\tilde{A}(\underline{u}|v_2)\tilde{B}(\underline{u}|v_1) = \frac{1}{b(v_2 - v_1)}\tilde{B}(\underline{u}|v_1)\tilde{A}(\underline{u}|v_2) - \frac{c(v_2 - v_1)}{b(v_2 - v_1)}\tilde{B}(\underline{u}|v_2)\tilde{A}(\underline{u}|v_1)$$

$$\tilde{A}^Q(\underline{u}|i)\tilde{B}(\underline{u}|v) = \frac{1}{b(u_i' - v)}\tilde{B}(\underline{u}'|v)\tilde{A}^Q(\underline{u}|i) - \frac{c(u_i' - v)}{b(u_i' - v)}\tilde{B}^Q(\underline{u}|i)\tilde{A}(\underline{u}|v)$$

$$\tilde{D}^\star(\underline{u}|v_2)\tilde{B}(\underline{u}|v_1) = -\frac{w(v_1 - v_2)}{b(v_1 - v_2)}\tilde{B}(\underline{u}|v_1)\tilde{D}^\star(\underline{u}|v_2) + \frac{c(v_1 - v_2)}{b(v_1 - v_2)}\tilde{B}(\underline{u}|v_2)\tilde{D}^\star(\underline{u}|v_1)$$

$$\tilde{D}^{\star Q}(\underline{u}|i)\tilde{B}(\underline{u}|v) = -\frac{w(v - u_i')}{b(v - u_i)}\tilde{B}(\underline{u}'|v)\tilde{D}^{\star Q}(\underline{u}|i) + \frac{c(v - u_i)}{b(v - u_i)}\tilde{B}^Q(\underline{u}|i)\tilde{D}^\star(\underline{u}|v).$$

Due to supersymmetry these relations are structurally different from those for the ordinary group case [10]. As a consequence the function $\tilde{\tau}$ has to satisfy a slightly modified functional equation (20) compared to τ in (17).

Next we act with the shift operator $\tilde{Q}^{1\cdots m}(\underline{u}|i) = \tilde{A}^{Q}(\underline{u}|i) + \tilde{D}^{\star Q}(\underline{u}|i)$ on the Bethe ansatz vector $\tilde{f}^{1\cdots m}(\underline{u})$ and repeat the arguments given above. Equations (2) are equivalent for all $i = 1, \ldots, m$, so we will restrict ourselves to $i = m$. Using the relations $\tilde{A}^{Q}(\underline{u}|m)\tilde{\Omega}^{1\cdots m} = \tilde{\Omega}^{1\cdots m}$ and $\tilde{D}^{\star Q}(\underline{u}|m)\tilde{\Omega}^{1\cdots m} = 0$ we obtain the representations ($\underline{u}' = \underline{u} + \xi \underline{e}_m$)

$$\tilde{A}^{\star Q}(\underline{u}|m)\tilde{f}^{1\cdots m}(\underline{u}) = \tilde{f}^{1\cdots m}(\underline{u}') + \sum_{\underline{v}}\sum_{i=1}^{k}\tilde{\Lambda}_A^{(i)}(\underline{u}|\underline{v})\tilde{B}^{Q}(\underline{u}|m)\tilde{B}(\underline{u}|v_k)\cdots\widehat{\tilde{B}(\underline{u}|v_i)}\cdots\tilde{B}(\underline{u}|v_1)\tilde{\Omega}\tilde{g}(\underline{u}|\underline{v}) \tag{21}$$

$$\tilde{D}^{\star Q}(\underline{u}|m)\tilde{f}^{1\cdots m}(\underline{u}) = \sum_{\underline{v}}\sum_{i=1}^{k}\tilde{\Lambda}_D^{(i)}(\underline{u}|\underline{v})\tilde{B}^{Q}(\underline{u}|m)\tilde{B}(\underline{u}|v_k)\cdots\widehat{\tilde{B}(\underline{u}|v_i)}\cdots\tilde{B}(\underline{u}|v_1)\tilde{\Omega}\tilde{g}(\underline{u}|\underline{v}). \tag{22}$$

By similar arguments to those given previously one can show that the functions $\tilde{\Lambda}_A^{(i)}(\underline{u}|\underline{v})$ and $\tilde{\Lambda}_D^{(i)}(\underline{u}|\underline{v})$ are given by

$$\tilde{\Lambda}_A^{(i)}(\underline{u}|\underline{v}) = -\frac{c(u'_m - v_i)}{b(u'_m - v_i)}\prod_{l<i}\frac{1}{b(v_i - v_l)}\prod_{l>i}\frac{-1}{b(v_l - v_i)}$$

$$\tilde{\Lambda}_D^{(i)}(\underline{u}|\underline{v}) = \frac{c(v_i - u_m)}{b(v_i - u_m)}\prod_{l<i}\frac{1}{b(v_i - v_l)}\prod_{l>i}\frac{-1}{b(v_l - v_i)}\prod_{l=1}^{m}b(u_l - v_i).$$

We made use of the fact that $w(\theta)w(-\theta) = 1$ and $w(\theta)/b(\theta) = -1/b(-\theta)$.

Again the 'wanted' contributions already guarantee the validity of (2) while a straightforward calculation using (20) and $c(x)/b(x) = -c(-x)/b(-x)$ shows that the 'unwanted' contributions sum up to zero. The $\tilde{R}$-symmetry is implied by the symmetry of $\tilde{g}(\underline{u}|\underline{v})$ in the variables $u_1, \ldots, u_m$ and the property $\tilde{R}_{ij}(\theta)\tilde{\Omega}^{1\cdots m} = \tilde{\Omega}^{1\cdots m}$.

Finally, we have proved that $f^{1\cdots n}$ given by the Bethe ansatz (16) solves the combined system of R-symmetry (1) and the matrix difference equations (2) if analogous relations hold for $\tilde{f}^{1\cdots m}$. It was shown that solutions to the dimensional reduced problem can be constructed explicitly by use of the Bethe ansatz (19).

3. Highest-weight property

We now investigate the $sl(2, 1)$ properties of the shift operator $Q^{1\cdots n}$ and of the solutions (16) constructed above. The behaviour $R_{ab}(x) = \Sigma_{ab} + (K/x)P_{ab} + \mathrm{O}(x^{-2})$ for $x \to \infty$ implies the asymptotic expansion

$$[T_{1\cdots n,a}(\underline{x}|u)]_{\alpha,\{\gamma\}}^{\beta,\{\nu\}} = \left[\Sigma_{1a}\cdots\Sigma_{na} + \frac{K}{u}\sum_{j=1}^{n}\Sigma_{1a}\cdots P_{ja}\cdots\Sigma_{na}\right]_{\alpha,\{\gamma\}}^{\beta,\{\nu\}} + \mathrm{O}(u^{-2})$$
$$= \sigma_{\alpha,\{\mu\}}\delta_\alpha^\beta\delta_{\mu_1}^{\nu_1}\cdots\delta_{\mu_n}^{\nu_n} + \frac{K}{u}\sigma_{\beta\alpha}\sigma_{\beta,\{\nu\}}M_{\alpha,\{\mu\}}^{\beta,\{\nu\}} + \mathrm{O}(u^{-2}).$$

The operators $M_{\alpha,\{\mu\}}^{\beta,\{\nu\}}$ have the form

$$M_{\alpha,\{\mu\}}^{\beta,\{\nu\}} = \sum_j \sigma_{\beta\nu_{j+1}}\cdots\sigma_{\beta\nu_n}\sigma_{\alpha\nu_{j+1}}\cdots\sigma_{\alpha\nu_n}\delta_{\mu_1}^{\nu_1}\cdots\delta_{\mu_{j-1}}^{\nu_{j-1}}\delta_{\mu_j}^{\beta}\delta_\alpha^{\nu_j}\delta_{\mu_{j+1}}^{\nu_{j+1}}\cdots\delta_{\mu_n}^{\nu_n}. \tag{23}$$

From this one derives the commutation relations

$$M_\alpha^{\alpha'} T^{\star}{}_\beta^{\beta'}(u) - \sigma_{\alpha\beta}\sigma_{\alpha\beta'}\sigma_{\alpha'\beta}\sigma_{\alpha'\beta'} T^{\star}{}_\beta^{\beta'}(u) M_\alpha^{\alpha'} = \delta_\beta^{\alpha'} T^{\star}{}_\alpha^{\beta'}(u) - \sigma_{\alpha\beta}\sigma_{\alpha\beta'}\sigma_{\alpha'\beta}\sigma_{\alpha'\beta'} \delta_\alpha^{\beta'} T^{\star}{}_\beta^{\alpha'}(u) \qquad (24)$$

where the quantum space indices have been neglected. A further consequence is

$$M_\alpha^{\alpha'} M_\beta^{\beta'} - \sigma_{\alpha\beta}\sigma_{\alpha\beta'}\sigma_{\alpha'\beta}\sigma_{\alpha'\beta'} M_\beta^{\beta'} M_\alpha^{\alpha'} = \delta_\beta^{\alpha'} M_\alpha^{\beta'} - \sigma_{\alpha\beta}\sigma_{\alpha\beta'}\sigma_{\alpha'\beta}\sigma_{\alpha'\beta'} \delta_\alpha^{\beta'} M_\beta^{\alpha'} \qquad (25)$$

for $u \to \infty$. This implies that the operators $M_\alpha^{\alpha'}$ are generators of $sl(2, 1)$ in the Cartan–Weyl basis (see [3, 24]). From (24) one can derive the invariance property $[M_\alpha^{\alpha'}, Q(\underline{u}|i)]_- = 0$. This means that from any solution of (2) further solutions may be constructed by applying raising and lowering operators of $sl(2, 1)$. The operators $W_\alpha = M_\alpha^\alpha$ (no summation with respect to α) satisfy the commutation relations $[W_\alpha, W_\beta]_- = 0$ and generate the Cartan subalgebra. For $\alpha = \beta$ the statistic signs in (23) cancel and therefore we obtain

$$[W_\alpha]_{\{\mu\}}^{\{\nu\}} = \sum_j \delta_{\mu_1}^{\nu_1} \cdots \delta_{\mu_{j-1}}^{\nu_{j-1}} \delta_{\mu_j}^{\alpha} \delta_\alpha^{\nu_j} \delta_{\mu_{j+1}}^{\nu_{j+1}} \cdots \delta_{\mu_n}^{\nu_n}. \qquad (26)$$

The highest-weight property of the Bethe ansatz functions $M_\alpha^{\alpha'} f^{1\cdots n}(\underline{x}) = 0$ for $\alpha' > \alpha$ is proven in a way analogous to that used in section 2. In other words, one uses commutation relations implied by (24), then commutes the matrices $M_\alpha^{\alpha'}$ through all B-operators to the right and finally, one uses certain eigenvalue equations. Again one has 'wanted' and 'unwanted' contributions and the summation guarantees the vanishing of the latter (cf [10]). After some calculation one obtains the weight vector which is defined by $W_\alpha f(\underline{x}) = w_\alpha f(\underline{x})$ and reads

$$\underline{w} = (n - m, m - k, k).$$

The highest-weight conditions are $w_1 \geqslant w_2 \geqslant -w_3$ and $w_1, w_2, w_3 \geqslant 0$ [3].

4. Conclusions and outlook

In this paper we have discussed a combined system of matrix difference equations based on the supersymmetric Lie algebra $sl(2, 1)$. Solutions are constructed by means of a nested version of the so-called off-shell Bethe ansatz and shown to be of highest weight with respect to $sl(2, 1)$. Due to the invariance of the shift operator $Q^{1\cdots m}$ under the generators of $sl(2, 1)$ it is possible to construct and classify further solutions by purely group-theoretic considerations.

It would be interesting to see whether there is a quantum integrable (relativistic) field theory associated with the supersymmetric t-J model. In that case the methods presented here could be used to determine the corresponding correlation functions. In this context the extension of our results to the q-deformed case $U_q[sl(2, 1)]$ would also be of interest [2, 25]. Recently, an integrable quantum field theory has been discussed which is based on the $osp(2, 2)$ graded Lie algebra [4] which is isomorphic to $sl(2, 1)$.

Acknowledgments

The author would like to thank M Karowski, R Schrader and, in particular, A Zapletal for numerous helpful and stimulating discussions as well as Z Maassarani for pointing out references [4, 7, 8] to him. He is also grateful to Studienstiftung des deutschen Volkes for support. The work was partially supported by DFG, Sonderforschungsbereich 288 'Differentialgeometrie und Quantenphysik'.

References

[1] Korepin V E and Eßler F H 1994 *Exactly Solvable Models of Strongly Correlated Electrons (Advanced Series in Mathematical Physics vol 18)* (Singapore: World Scientific)
[2] Foerster A and Karowski M 1993 *Nucl. Phys.* B **408** 512
[3] Foerster A and Karowski M 1993 *Nucl. Phys.* B **396** 611
[4] Bassi Z S and LeClair A 1999 *Preprint* hep-th/9911105
Bassi Z S and LeClair A 1999 *Preprint* CLNS 99/1646
[5] Göhmann F and Korepin V E 2000 *J. Phys. A: Math. Gen.* **33** 1199
Göhmann F and Korepin V E 1999 *Preprint* hep-th/9910253
[6] Pfannmüller M P and Frahm H 1996 *Nucl. Phys.* B **479** 575
[7] Ramos P B and Martins M J 1996 *Nucl. Phys.* B **474** 678
[8] Maassarani Z 1995 *J. Phys. A: Math. Gen.* **28** 1305
[9] Hinrichsen H and Rittenberg V 1992 *Phys. Lett.* B **275** 350
[10] Babujian H, Karowski M and Zapletal A 1997 *J. Phys. A: Math. Gen.* **30** 6425
[11] Zapletal A 1998 *J. Phys. A: Math. Gen.* **31** 9593
[12] Babujian H, Fring A, Karowski M and Zapletal A 1999 *Nucl. Phys.* B **538** 535
[13] Smirnov F A 1992 *Form Factors in Completely Integrable Models of Quantum Field Theory (Advanced Series in Mathematical Physics vol 14)* (Singapore: World Scientific)
[14] Tarasov V and Varchenko A 1997 *Invent. Math.* **128** 501
[15] Frenkel I B and Reshetikhin N Y 1992 *Commun. Math. Phys.* **146** 1
[16] Berg B, Karowski M and Weisz P 1979 *Phys. Rev.* D **19** 2477
[17] Reshetikhin N Y 1992 *Lett. Math. Phys.* **26** 153
[18] Knizhnik V G and Zamolodchikov A B 1984 *Nucl. Phys.* B **247** 83
[19] Faddeev L D and Takhtadzhyan L A 1981 *Zap. Nauch. Semin. LOMI* **109** 134
[20] Babujian H 1990 *Proc. 24th Int. Symp.* (Ahrenshoop: Zeuthen)
[21] Babujian H 1993 *J. Phys. A: Math. Gen.* **26** 6981
[22] Korotkin D and Nicolai H 1996 *Nucl. Phys.* B **475** 397
[23] Bracken A J, Gould M D and Zhang R B 1990 *Mod. Phys. Lett.* A **5** 831
[24] Scheunert M, Nahm W and Rittenberg V 1977 *J. Math. Phys.* **18** 155
[25] Foerster A 1996 *J. Phys. A: Math. Gen.* **29** 7625